1) $|\Psi\rangle = \sum_{r,s} c_{rs}|r\rangle|s\rangle$

$\rho_t = |\Psi\rangle\langle\Psi|$ describes the system plus reservoir

We find ρ describing just the system by tracing out the reservoir,

$$\rho = Tr_{res}(\rho_t) = \sum_{r'} \langle r'|\Psi\rangle\langle\Psi|r'\rangle$$

$$= \sum_{r'} \langle r'|\left(\sum_{rs} c_{rs}|r\rangle|s\rangle\right)\left(\sum_{r''s''} c^*_{r''s''}\langle r''|\langle s''|\right)|r'\rangle$$

$$= \sum_{r}\left[\left(\sum_{s} c_{rs}|s\rangle\right)\left(\sum_{s'} c^*_{rs'}\langle s'|\right)\right] = \sum_{ss'}\underbrace{\left[\sum_{r} c_{rs}c^*_{rs'}\right]}_{\rho_{ss'}}|s\rangle\langle s'|$$

Now evaluating the trace over the system,

$$Tr_{sys}(\rho) = \sum_{s}\langle s|\rho|s\rangle$$

$$= \sum_{s}\sum_{r}\langle s|\left(\sum_{s'} c_{rs'}|s'\rangle\right)\left(\sum_{s''} c^*_{rs''}\langle s''|\right)|s\rangle$$

$$= \sum_{s}\sum_{r} c_{rs}c^*_{rs} = 1$$ by normalization of $|\Psi\rangle$.

2) The expression for the one-particle density matrix,

$$\rho(\vec{x},\vec{x}') = \frac{N_0(T)}{N}\cdot\phi_n(\vec{x})\,\phi_n^*(\vec{x}') + \frac{1}{N}\int\frac{d^3k}{(2\pi)^3}\frac{e^{i\vec{k}\cdot\vec{x}}\,e^{-i\vec{k}\cdot\vec{x}'}}{e^{\beta(\epsilon_k-\mu)}-1}$$

So to find the correction we evaluate the second term.

First, let us move to spherical coords with the z-axis along $(\vec{x}-\vec{x}')$,

$$\int d^3\vec{k}\,\frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}{e^{\beta(\epsilon_k-\mu)}-1} = \int_0^{2\pi}d\phi\int_{-1}^{1}d(\cos\theta)\int_0^\infty k^2dk\,\frac{e^{ik|\vec{x}-\vec{x}'|\cos\theta}}{e^{\beta(\epsilon_k-\mu)}-1}$$

$$= 2\pi\int_0^\infty dk\,\frac{k^2}{e^{\beta(\epsilon_k-\mu)}-1}\int_{-1}^{1}dz\,e^{ik|\vec{x}-\vec{x}'|z}$$

$$= 2\pi\int_0^\infty dk\,\frac{k^2}{e^{\beta(\epsilon_k-\mu)}-1}\,\frac{\left(e^{ik|x-x'|}-e^{-ik|x-x'|}\right)}{ik|\vec{x}-\vec{x}'|}$$

$$= \frac{4\pi}{|\vec{x}-\vec{x}'|}\int_0^\infty dk\,\frac{k\sin(k|\vec{x}-\vec{x}'|)}{e^{\beta(\epsilon_k-\mu)}-1}$$

$$= \frac{4\pi}{|\vec{x}-\vec{x}'|}\,\mathrm{Im}\int_{-\infty}^{\infty}dk\,\frac{k\,e^{ik|\vec{x}-\vec{x}'|}}{e^{\beta(\epsilon_k-\mu)}-1}$$

For a single particle, $\epsilon_k = \frac{\hbar^2k^2}{2m}$, and in this limit we can set $z\to 1$,

$$= \frac{4\pi}{|\vec{x}-\vec{x}'|}\,\mathrm{Im}\int_0^\infty dk\,\frac{k\,e^{ik|\vec{x}-\vec{x}'|}}{e^{\beta\frac{\hbar^2k^2}{2m}}-1}$$

Note the integrand, or rather the imaginary part which is all we're interested in, even in k,

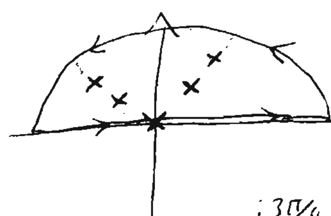
$$= \frac{2\pi}{|\vec{x}-\vec{x}'|}\,\mathrm{Im}\int_{-\infty}^{\infty}dk\,\frac{k\,e^{ik|\vec{x}-\vec{x}'|}}{e^{\beta\hbar^2k^2/2m}-1}$$

At this point we evaluate this integral by contour integration.

The poles are at $e^{\beta\frac{\hbar^2k^2}{2m}}-1=0 \Rightarrow \frac{\beta\hbar^2k^2}{2m}=\pm 2\pi n i$ for $n\in\mathbb{N}$

We close the contour in the upper half plane,

The residue at zero is on the contour so we take the principal value.



So,

$$= \frac{2\pi}{|\vec{x}-\vec{x}'|}\,\mathrm{Im}\left(\frac{2\pi i}{2}\mathrm{Res}(k=0) + \sum_{n=1}^{\infty}2\pi i\left[\mathrm{Res}(k=\sqrt{2\pi n}\,e^{i\pi/4}) + \mathrm{Res}(k=\sqrt{2\pi n}\,e^{i3\pi/4})\right]\right)$$

$$\mathrm{Res}(k=0) = \left.\frac{k^2e^{ik|\vec{x}-\vec{x}'|}}{e^{\beta\hbar^2k^2/2m}-1}\right|_{k=0} = \frac{2m}{\beta\hbar^2}$$

$\mathrm{Res}(k=e^{i\pi/4}\sqrt{2\pi n}) \propto e^{-(|\vec{x}-\vec{x}'|\sqrt{\pi n})}$, since we are interested in the limit of large $|\vec{x}-\vec{x}'|$, we can drop these residues. Therefore to first order,

$$\frac{2\pi}{\ldots}\quad 2\pi m \Rightarrow \rho(\vec{x},\vec{x}') \simeq \rho_0 + \frac{m}{2\pi\beta\hbar^2}\,\frac{1}{N|\vec{x}-\vec{x}'|}$$

3a) $n = \frac{N}{V} = 2\int_0^{k_F} \frac{d^3k}{(2\pi)^3} = \frac{k_F^3}{3\pi^2} \Rightarrow k_F = (3n\pi^2)^{1/3}$

$n = \frac{N}{V} = \left(\frac{M}{2m_p}\right) / \left(\frac{4}{3}\pi R^3\right) = \frac{3M}{8\pi R^3 m_p}$

$k_F = \left(\frac{9\pi^2 M}{8 m_p \pi R^3}\right)^{1/3} = \left(\frac{9\pi M}{8 m_p}\right)^{1/3} \frac{1}{R}$ $\qquad \mathcal{E}_F = \frac{\hbar^2}{2m_e R^2}\left(\frac{9}{8}\right.$

b) $\mathcal{E} = 2V\int_0^{k_F} \frac{d^3k}{(2\pi)^3} \frac{\hbar^2 k^2}{2m_e} = \frac{\hbar^2}{m_e} \frac{4\pi}{(2\pi)^3} \int_0^{k_F} k^4 dk$

$= \frac{\hbar^2 V}{2\pi^2 m_e} \frac{k_F^5}{5} = \mathcal{E}_F \cdot \frac{V k_F^3}{10\pi^2}$

$\mathcal{E}_F \equiv \frac{\hbar^2 k_F^2}{2m_e}$, $\quad V k_F^3 = V(3\frac{N}{V}\pi^2) = 3N\pi^2$ $\qquad N = \frac{M}{2m_p}$

$= \frac{3N}{10}\mathcal{E}_F \qquad = \frac{3M}{20m_p}\frac{\hbar^2}{2m_e}\left(\frac{9\pi M}{8m_p}\right)^{2/3}\frac{1}{R^2}$

c) $U_g = -\frac{3GM^2}{5R}$

d) $\mathcal{E} = \frac{3N}{10}\mathcal{E}_F - \frac{3GM^2}{5R} = \left(\frac{\hbar^2}{2m_e}\left(\frac{9\pi M}{8m_p}\right)^{2/3}\frac{1}{R^2}\right)\cdot\frac{3M}{20m_p} - \frac{3GM^2}{5R}$

$\frac{\partial \mathcal{E}}{\partial R} = \frac{3M\hbar^2}{40m_p m_e}\left(\frac{9\pi M}{8m_p}\right)^{2/3}\left(\frac{-2}{R^3}\right) + \frac{3GM^2}{5R^2} = 0$

$\Rightarrow \quad \frac{3GM^2}{5}R = \frac{3M\hbar^2}{20m_p m_e}\left(\frac{9\pi M}{8m_p}\right)^{2/3}$

$R = \frac{\hbar^2}{4GMm_p m_e}\left(\frac{9\pi M}{8m_p}\right)^{2/3}$

$= \left(\frac{9\pi}{8}\right)^{2/3}\frac{\hbar^2}{4GM^2 m_e}\left(\frac{M}{m_p}\right)^{5/3}$

e) $\varepsilon_k = \hbar c k$

$$\varepsilon = \langle KE \rangle = 2V \int_0^{k_f} \frac{d^3k}{(2\pi)^3} (\hbar c k)$$

$$= \frac{V \hbar c}{2\pi^2} \int_0^{k_f} dk\, k^3 = \frac{\frac{4}{3}\pi R^3 \hbar c}{2\pi^2} \frac{k_f^4}{4}$$

$$= \frac{\hbar c}{6\pi} R^3 \left[\left(\frac{9\pi M}{8 m_p} \right)^{1/3} \frac{1}{R} \right]^4$$

$$= \frac{\hbar c}{6\pi} \left(\frac{9\pi M}{8 m_p} \right)^{4/3} \cdot \frac{1}{R}$$, same $\frac{1}{R}$ dependence as U_g.

f) $\varepsilon > |U_g|$ or collapse $(R \to 0)$

$$\Rightarrow \frac{\hbar c}{6\pi} \left(\frac{9\pi M}{8 m_p} \right)^{4/3} > \frac{3GM^2}{5}$$

Setting equal to find the limit,

$$M^2 < \frac{5\hbar c}{18\pi G} \left(\frac{9\pi M}{8 m_p} \right)^{4/3}$$

$$M < \left(\frac{5\hbar c}{18\pi G} \right)^{3/2} \left(\frac{9\pi}{8 m_p} \right)^2$$

4) $\varepsilon_k = -2t\cos(ka)$ $k = [-\frac{\pi}{a}\ \ \frac{\pi}{a}]$

a) To find the number of states in the band,

$$N = \sum_k (2 \text{ spin states}) = 2\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \left(\frac{L}{2\pi}\partial k\right) = \frac{2L}{a}$$

So a half filled band,

$$n = \frac{N\cdot\frac{1}{2}}{L} = \frac{1}{a}$$

b) The band splits in half at $k_F = 0$, so $\varepsilon_F = 0$.

c) $$\frac{\partial \varepsilon}{\partial k} = 2at\sin(ka)$$

$$\partial\varepsilon = 2at\sin(ka)\,\partial k \qquad \rho(k) = \frac{L}{2\pi}$$

So,

$$\rho(k)\partial k = \frac{L}{2\pi}\frac{1}{2at\sin(ka)}\partial\varepsilon = \frac{L}{2\pi}\frac{1}{2at\sqrt{1-\cos^2(ka)}}\partial\varepsilon$$

$$= \frac{L}{2\pi}\frac{1}{2at\sqrt{1-(\varepsilon/2t)^2}}\partial\varepsilon$$

$$= \frac{L}{2\pi a}\frac{1}{\sqrt{4t^2-\varepsilon^2}}\partial\varepsilon$$

$$\Rightarrow \rho(\varepsilon) = \frac{L}{2\pi a}\frac{1}{\sqrt{4t^2-\varepsilon^2}} \times 2 \leftarrow \text{spins}$$

d) First, for those looking for a more illuminating treatment of this sort of problem I recommend Ashcroft & Mermin, look under the entries for Sommerfeld expansion.

There is a quick symmetry argument here that gives the second result that μ is independent of T. The fermi function is symmetric around μ in the sense that $f(\mu+x) = 1 - f(\mu - x)$. When combined with the situation here where $\rho(\varepsilon)$ is symmetric around ε_F, you can see that raising the temperature will not change N, so μ does not need to change with T to enforce the constancy of N.

4e) $E = \int_{-2t}^{2t} \partial\varepsilon\, \rho(\varepsilon) f(\varepsilon)\, \varepsilon$

$$\rho(\varepsilon) = \frac{1}{\sqrt{(2t)^2 - \varepsilon^2}} \frac{L}{\pi a} \qquad f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon+\mu)} + 1} \qquad \mu = 0$$

$$\Rightarrow E = \frac{L}{\pi a} \int_{-2t}^{2t} \partial\varepsilon \frac{\varepsilon}{\sqrt{(2t)^2 - \varepsilon^2}} \frac{1}{e^{\beta\varepsilon} + 1}$$

We're going to use the Sommerfeld expansion, (Ashcroft + Mermin 2.70)

$$\int_{-\infty}^{\infty} H(\varepsilon) f(\varepsilon)\partial\varepsilon = \int_{-\infty}^{\mu} H(\varepsilon)\partial\varepsilon + \frac{\pi^2}{6}(k_BT)^2 H'(\mu) + \frac{7\pi^4}{360}(k_BT)^4 H'''(\mu) + O\left(\frac{k_B}{\mu}\right.$$

In our case,

$$H(\varepsilon) = \frac{\varepsilon}{\sqrt{(2t)^2 - \varepsilon^2}} \qquad \mu = 0$$

So now we need to evaluate the terms in the Sommerfeld expansion,

$$\int_{-\infty}^{\mu} H(\varepsilon)\partial\varepsilon = \int_{-2t}^{0} \partial\varepsilon \frac{\varepsilon}{\sqrt{(2t)^2 - \varepsilon^2}} \qquad x = 4t^2 - \varepsilon^2 \qquad \partial x = -2\varepsilon\partial\varepsilon$$

$$= -\frac{1}{2}\int_0^{4t^2} \partial x \frac{1}{\sqrt{x}} = -\frac{1}{2}\left(2\sqrt{x}\right|_0^{4t^2} = -2t$$

$$H'(\mu) = \frac{\partial}{\partial\varepsilon}\left(\frac{\varepsilon}{\sqrt{(2t)^2 - \varepsilon^2}}\right)\Big|_{\varepsilon=0} = \left(\frac{1}{\sqrt{4t^2 - \varepsilon^2}} - \frac{\varepsilon}{2}(4t^2 - \varepsilon^2)^{-3/2}(-2\varepsilon)\right|_{\varepsilon=0}$$

$$= \left((4t^2 - \varepsilon^2)^{-1/2} + \varepsilon^2(4t^2 - \varepsilon^2)^{-3/2}\right|_{\varepsilon=0}$$

$$= \frac{1}{2t}$$

$$H'''(\mu) = \frac{\partial^3}{\partial\varepsilon^3}\left(\frac{\varepsilon}{\sqrt{4t^2 - \varepsilon^2}}\right)_{\varepsilon=0} = \frac{\partial^2}{\partial\varepsilon^2}\left((4t^2 - \varepsilon^2)^{-1/2} + \varepsilon^2(4t^2 - \varepsilon^2)^{-3/2}\right)\Big|_{\varepsilon=0}$$

$$= \left[\frac{\partial}{\partial\varepsilon}\left(\varepsilon(4t^2 - \varepsilon^2)^{-3/2}\right) + 2(4t^2 - \varepsilon^2)^{-3/2} + \varepsilon f_1(\varepsilon) + \varepsilon^2 f_2(\varepsilon\right.$$

$$= \left[3(4t^2 - \varepsilon^2)^{-3/2} + \varepsilon f_1(\varepsilon) + \varepsilon^2(f_2(\varepsilon))\right|_{\varepsilon=0}$$

$$= 3(4t^2)^{-3/2} = \frac{3}{8t^3}$$

So, plugging this back in,

$$E = \frac{L}{\pi a}\left[-2t + \frac{\pi^2}{6}(k_BT)^2\frac{1}{2t} + \frac{7\pi^4}{360}(k_BT)^4\left(\frac{3}{8t^3}\right) + \ldots\right]$$

f) Specific heat is just the derivative of the above,

$$C_V = \frac{\partial E}{\partial T} = \frac{L}{\pi a}\left[\frac{\pi^2}{3}\frac{k_B^2}{2t}T + \frac{7\pi^4}{240}\frac{k_B^4}{t^3}T^3 + \ldots\right]$$