Physics 219 - Problem Set 1

Due Date: January 20, 2009

1. Stirling's formula. The Γ function is defined as:

$$\Gamma(z) = \int_0^\infty dt\, t^{z-1} e^{-t} \tag{1}$$

   (a) Show that $\Gamma(z+1) = z!$ for integers $z \geq 0$.

   (b) Using the method of steepest descents, derive the Stirling formula

$$\ln(N!) \approx N \ln N - N \tag{2}$$

   (c) What is the next subleading term?

2. Entropy of a Classical Ideal Gas in the Microcanonical Ensemble.

   (a) What is the surface area of a sphere in d dimensions as a function of its radius, r?

   (b) Consider a classical ideal gas of N particles of mass m in 3 dimensions. Their total energy is $E = \frac{1}{2m}\sum_{i=1}^{N} \mathbf{p}_i^2$, where $\mathbf{p}_i$ is the momentum of the i^{th} particle. What is the surface area of the $3N$-dimensional sphere of fixed energy?

   (c) What is the entropy of the gas, including the volume in position space?

   (d) What are the Temperature and Pressure of the gas, as a function of N and E?

3. Show that the specific heat at constant volume, $C_V = \left(\frac{\partial \langle E \rangle}{\partial T}\right)_V$ is proportional in the canonical ensemble to the mean-squared fluctuation of the energy, $\langle E^2 \rangle - \langle E \rangle^2$, and find the proportionality constant.

4. Violations of the second law. Consider two spin systems with N sites each and a spin-1/2 on each site. They are in the same magnetic field H, but are at different temperatures, T_1 and T_2 and, thus, two different magnetizations. Once the two spin blocks are put in contact, energy and magnetization are allowed to flow between the two blocks, and their value fluctuates. What is the probability that once we separate the two spin blocks, their entropy has decreased? Suppose that $N = 1000$, $H = 1$ T, $T_1 = 90$ K, and $T_2 = 100$ K. Also consider the case $N = 10^{23}$. Assume that the energy of a spin in a field is $E = -g\mu_B H S^z$

(we assume that the spins interact weakly enough that we can neglect their interaction in comparison to the magnetic field, but that they still interact enough to exchange energy and angular momentum), with $g = 2$, $\mu_B = 5.788 \times 10^{-5}$ eV/T, and $S^z = \pm\frac{1}{2}$. (Hint: first calculate the energies of the two blocks.)

of this gas.

you compute the energy as a function of temperature?

5. The von Neumann entropy, $S = -\sum_i p_i \log_2 p_i$, is used in information theory.

   (a) Consider a system C composed of two subsystems, A and B. Configurations of the system are specified by specifying the states (a,b) of the two subsystems. A configuration (a,b) has probability $p_{(a,b)}$ of occurring which is given by the product of independent probabilities for the two subsystems, $p_{(a,b)} = p_a p_b$. Show that the von Neumann entropy for C is given by the sum of the von Neumann entropies for the two subsystems, $S_C = S_A + S_B$.

   (b) How many bits of information can one deliver with a string of N letters, with letters are chosen from a set $\{a_1, a_2, \ldots, a_n\}$ if a_i appears with probability p_i? The information capacity per letter is cannel the capacity of the channel. The English language has 26 letters. Their average frequencies can be found at on the web (see, e.g. http://www.askoxford.com/asktheexperts/faq/aboutwords/frequency?view=uk). How much information is contained in an English word with 5 letters? What might be wrong about this estimate?