Physics 219 - Problem Set 2

Due Date: February 3, 2009

1. Equipartition Theorem.

   An ideal gas has a Hamiltonian that depends only on particle momenta,

   $$\mathcal{H} = \sum_{i=1}^{N} \frac{\vec{p}_i^2}{2m}$$

   where the momentum $\vec{p}_i$ is a d-dimensional vector. Consequently, each particle has average energy $\frac{d}{2} k_B T$.

   Consider a system with N particles with d-dimensional coordinates $\vec{x}_i$ and momenta $\vec{p}_i$, $i = 1, \ldots, N$. Show that if the Hamiltonian is:

   $$\mathcal{H} = \sum_{i=1}^{N} a\,|\vec{p}_i|^{n_p} + b\,|\vec{x}_i|^{n_x}$$

   Then the energy per particle is (convince yourself that it doesn't matter whether or not they are distinguishable) :

   $$\frac{E}{N} = d\left(\frac{1}{n_p} + \frac{1}{n_x}\right) k_B T$$

   This is the generalized equipartition theorem.

2. Consider a classical diatomic gas in three dimensions. We will ignore the vibrational degrees of freedom of the molecule and concentrate on its translational and rotational degrees of freedom. We will assume that it has total mass M and moment of inertia I about either of the two axes perpendicular to the axis joining the two atoms (i.e. the symmetry axis of the molecule). Assume that two configurations of the molecule which differ only by a rotation about its symmetry axis cannot be distinguished (i.e. ignore rotations about the molecule's symmetry axis). Also assume that the molecules are indistinguishable.

   (a) What is the specific heat of a gas of N such molecules at temperature T?

   (b) Continue to treat the translational motion classically, but now treat the rotational motion quantum-mechanically. Each molecule has rotational energy levels

   $$E = \frac{\hbar^2}{2I} j(j+1) \qquad j = 0, 1, \ldots$$

The angular momentum j state of the molecule has degeneracy $2j+1$. Write down an expression for the partition function, but don't attempt to evaluate it in this part.

(c) Evaluate the partition function in the low-temperature limit and thereby obtain the specific heat at low temperatures.

(d) Evaluate the partition function in the high-temperature limit and thereby recover the classical result of part (a).

3. Magnetic Cooling.

The magnetization and temperature are intimately related in spin systems. Debye (1926) and Giaque (1927) realized that this could be exploited for cooling.

Consider a gas of N Li atoms in a container of volume V. Each Lithium atom has an uncompensated $2s$ electron with magnetic dipole moment $\gamma S_z = g\mu_B S_z$, where $S_z = \pm\frac{1}{2}$ is the component of the spin in the z-direction. The gas is slowly cooled to temperature $T = 1$K in a field $H = 2$ Tesla. Then, the container is thermally insulated. The field H is slowly reduced to zero. This step is called adiabatic 'magnetic cooling'.

(Recall that $g = 2$, $\mu_B = 9.27 \cdot 10^{-24}$ J·T. The mass of a Lithium atom can be obtained from the periodic table.)

(a) What is the free energy of the gas as a function of T, V, N, and H? Hint: the partition function factorizes into the product of the spin partition function and the kinetic partition function. You may treat the kinetic partition function classically, as in problem (2).

(b) What is the entropy of the gas as a function of T, V, N, and H?

(c) What remains constant as the magnetic field is adiabatically reduced to zero?

(d) What is the temperature at the end of the process?

(e) If we could start with an arbitrarily large magnetic field H, what would be the maximum factor by which we could decrease the temperature of the gas?

4. 'Faux-tons.'

Consider a system of light-like particles in two dimensions with energy-wavevector relation $\omega = ak^3$. What is the specific heat at low temperatures?

5. Einstein Phonons.

   In addition to the ‘acoustic phonons’, with $\omega(k) \propto k$ at small k, which we studied in class, solids also have ‘optical phonons’. Optical phonons (as well as electrons) are responsible for the interactions of a solid with light because they have higher frequencies. One model for the dispersion relation for optical phonons is $\omega(k) = \omega_E$, where ω_E is some constant independent of k. For simplicity, you may assume that the solid is a cubic lattice of lattice constant a.

   Compute the contribution from optical phonons to the specific heat of a solid as a function of temperature.