Physics 219 - Problem Set 3

Due Date: February 17, 2009

1. When electrons are confined to move in a two-dimensional plane (e.g. in a semiconductor quantum well or heterostructure), they will form a triangular lattice if their density is too low. Such a state is called a *Wigner crystal*. The transverse vibrations of such a crystal are similar to those of the ionic lattice, but the longitudinal vibrations satisfy $\omega_{\vec{k}} = a\sqrt{k}$ as a result of the unscreened long-ranged Coulomb interaction between electrons. What is the contribution to the low-temperature specific heat of a Wigner crystal coming from longitudinal vibrations.

2. In addition to the equality of mixed partials, there are some other mathematical identities which are useful in deriving thermodynamic relations. It is possible to relate a partial with w held constant to one with u held constant:

$$\left(\frac{\partial f}{\partial x}\right)_w = \left(\frac{\partial f}{\partial x}\right)_u + \left(\frac{\partial f}{\partial u}\right)_x \left(\frac{\partial u}{\partial x}\right)_w$$

It is also possible to relate a partial derivative with f held constant to derivatives of f with respect to variables x, t:

$$\left(\frac{\partial x}{\partial t}\right)_f = \frac{\left(\frac{\partial f}{\partial t}\right)_x}{\left(\frac{\partial f}{\partial x}\right)_t}$$

Finally,

$$\left(\frac{\partial x}{\partial f}\right)_u = \frac{1}{\left(\frac{\partial f}{\partial x}\right)_u}$$

Using these identities, show that

$$\left(\frac{\partial p}{\partial V}\right)_S = -\frac{C_p}{C_V\, V\, \kappa_T}$$

where the compressibility, κ_T, is given by:

$$\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T$$

3. In this problem, we will consider the statistical mechanics of chemical reactions. We will focus, for simplicity, on reactions of the form $X + X \leftrightarrow X_2$.

(a) We will need the *intrinsic* chemical potentials of monatomic and diatomic gases as a function of their atomic masses, temperature, and concentration. By intrinsic, we mean the bare chemical potential, due only to their translational and rotational degrees of freedom, excluding potential energies. Assume, as in problem 2 of the problem set 2, that the rotational energy of a diatomic molecule is of the form

$$H_{\text{rot}} = \frac{1}{2I}\left(L_1^2 + L_2^2\right)$$

where I is the moment of inertia about either of the two axes perpendicular to the axis joining the two atoms (i.e. the symmetry axis of the molecule). L_1 and L_2 are the angular momenta along these two axes. m_X and m_{X_2} are the masses of an X atom and a diatomic molecule.

What are the chemical potentials of the monatomic and diatomic gases as a function of density and temperature?

(b) When two X atoms bind into a molecule, a binding energy ε is released: $X + X \to X_2 + \varepsilon$. What are the total chemical potentials of X atoms and X_2 molecules, including both their intrinsic chemical potentials and binding energies?

(c) Show that if the chemical reaction occurs in a container of fixed volume and at constant temperature, then

$$dF = \mu_X dN_X + \mu_{X_2} dN_{X_2}$$

(d) If the total number of X atoms in the container (in both monatomic and diatomic form) is n, what is the relation between n, n_X, and n_{X_2}? What is the relationship between dN_X and $\mu_{X_2} dN_{X_2}$?

(e) By minimizing F with respect to n_X, solved for n_X and n_{X_2} in terms of n, T, I, m_{X_2}, and m_X.

4. Consider a gas of bosonic particles with dispersion

$$\varepsilon_{\vec{k}} = a k^s$$

in d-dimensions.

(a) What is the minimum (integer) dimension in which BEC occurs?

(b) At the lowest (integer) dimension in which BEC occurs, how does T_{BEC} depend on the density of the gas?

(c) Suppose, instead, that the dispersion of the gas is:

$$\varepsilon_{\vec{k}} = \frac{1}{2a}\left(k^2 - k_0^2\right)^2$$

where k_0 is a constant. Can such a gas Bose condense?

5. Show that for an ideal Bose gas,

$$\frac{C_P - C_V}{Nk_B} = \left(\frac{C_V}{\frac{3}{2}Nk_B}\right)^2 \frac{g_{1/2}(z)}{g_{3/2}(z)}$$

What happens at $T < T_c$?