Physics 219 - Problem Set 4

Due Date: March 3, 2009

1. Density Matrices.

   (a) Consider a system + reservoir in a pure state $|\psi\rangle$, as in class:

   $$|\psi\rangle = \sum_{r,s} c_{r,s} |r\rangle |s\rangle$$

   where $|r\rangle$ describes the state of the reservoir and $|s\rangle$ describes the state of the system. The system by itself will be described by a density matrix ρ. Show that ρ satisfies $\mathrm{tr}(\rho) = 1$.

   (b) Following the logic which we used in the case of the canonical ensemble, show that the von Neumann entropy of a system described by density matrix ρ is

   $$S = -k_B \,\mathrm{tr}(\rho \ln \rho)$$

   Show that this result is basis-independent.

2. One-particle Density Matrix.

   In class, we deduced an expression for the one-particle density matrix of an ideal Bose gas for $T < T_c$. From this expression, determine the leading correction to $\rho(x, x') = N_0(T)/N$ in the limit of large $|x - x'|$.

3. White Dwarves.

   White dwarves are stars which are abnormally faint because they have used up most of their hydrogen. We can think of them as consisting only of Helium. However, their temperature $\sim 10^7$ K is high enough that the Helium is completely ionized; the star can, thus, be viewed as N electrons and $N/2$ Helium nuclei. Chandrasekhar conjectured that such a star's radius is determined by a balance between the degeneracy pressure of the electrons and the gravitational pull of the Helium.

   The temperature of the star, $\sim 10^7$ K, though high enough to ionize the Helium, is still much smaller than the Fermi energy of the electrons in the star. The mass M of the star is due almost entirely to Helium, so $M \approx (N/2) m_{\mathrm{He}} = 2N m_p$, where m_p is the mass of a proton.

(a) If the radius of the star is R, what is the Fermi energy of the electrons in it? Assume the electrons are non-relativistic. Express your answer in terms of M and R, and whatever other physical constants you deem necessary.

(b) What is the total kinetic energy of the electron gas?

(c) What is the gravitational energy of the star?

(d) Minimize the total energy (gravitational + electronic) to find the equilibrium volume of the star. What is this radius? What is the Fermi energy of the electrons? How does it compare to the rest mass energy of the electrons (mc^2)? Typically, the mass of white dwarves is of order of the suns mass, 10^{30} kg. Use this to obtain numerical values in this part only.

(e) From 5d, you should infer that the electrons are actually not adequately described in terms of non-relativistic mechanics. Assume the opposite limit, of ultra-relativistic electrons, with kinetic energy given by $\varepsilon_k = \hbar ck$. What is the energy of the gas assuming the electrons are ultra-relativistic? Show that this energy has the same dependence on the radius of the star as the gravitational energy.

(f) From the conditions of stability infer a condition on the mass of a star such that it would form a white dwarf, rather than collapse. The critical mass you find here is the Chanderasekhar mass. The refined result is $M_C = 1.44 M_{\text{sun}}$.

4. One-dimensional Metal.

Consider the following one-dimensional toy-model of a metal. The dispersion is

$$\varepsilon_k = -2t \cos ka$$

Here, t is an energy scale (typically in the electron volt range) and a is the lattice constant. The momentum is in the range $k \in [-\pi/a, \pi/a]$. Note that the minimum energy of an electron is -2t, not zero. Let us suppose that the band is half full, i.e. half of the states of the band are empty and half are filled

(a) What is the electronic density (in electrons per unit length)?

(b) What is the Fermi energy?

(c) What is the density of states $\rho(\varepsilon)$?

(d) What is the chemical potential as a function of temperature up to fourth order in T?

(e) What is the energy up to fourth order in T?

(f) What is the specific heat up to third order in T?